Analytic Functions and Ordinary Differential Equations in Exact Real Arithmetic

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[1] Joint work with Akitoshi Kawamura and Florian Steinberg

Computable Real Numbers

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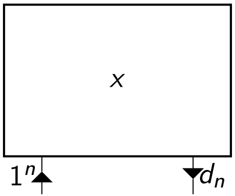

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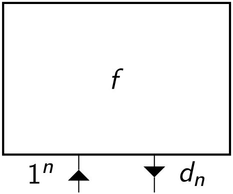

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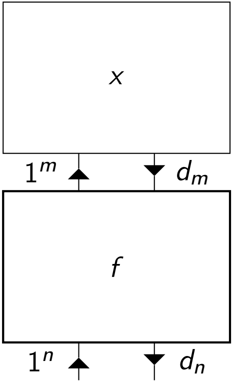

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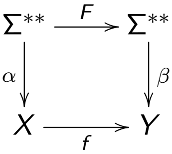

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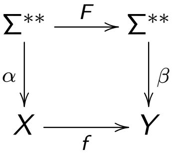

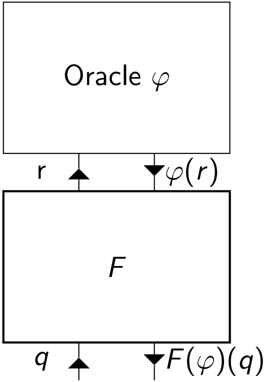

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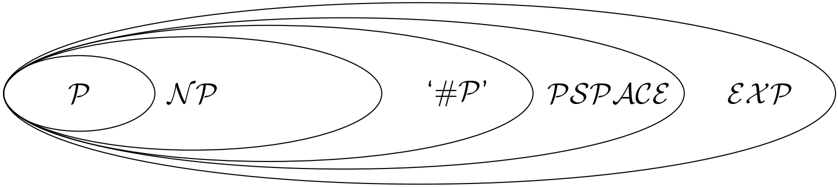
- In the end, the user can output an approximation to the result of the computation with any desired precision
- Many predefined functions already exist (sine,...)
- New real numbers and functions can be defined (limit operator)

Complexity Theory



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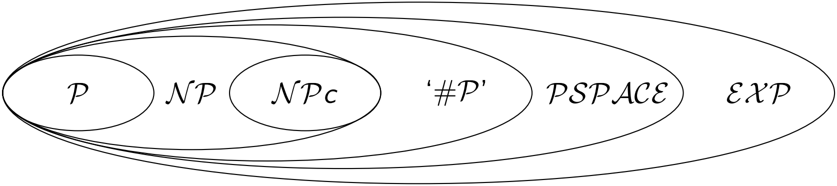

Data-types for functions

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Fact

For general polynomial time computable functions, many important operators have been shown to be computationally hard. For example

- *Polynomial time computable functions may have noncomputable derivatives. (Ko 1983)*
- *Parametric maximization is NP-hard. (Ko/Friedman (1982))*
- *Integration is #P-hard. (Friedman (1984))*

Analytic Function

An analytic function is a function locally given by a complex power series.

Definition (Analytic Function)

$f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ is analytic if for any $x_0 \in D$ the Taylor-series

$$T(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges to $f(x)$ for x in a neighborhood of x_0.

Some non-uniform results

$$a_m = \frac{f^{(m)}(x_0)}{m!}, \quad f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^k \quad \text{for } x \in B(x_0, R)$$

Theorem (Pour-El, Richards, Ko, Friedman, Müller (1987/1989))

f is (polytime) computable iff $(a_m)_{m \in \mathbb{N}}$ is.

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From that polynomial time computability of the derivative and the anti-derivative of a function follows immediately.

Some non-uniform results

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Theorem (Müller (1995))

- *The operator* $f \to (a_m)_{m\in\mathbb{N}}$ *is not computable.*
- *The evaluation operator* $((a_m)_{m\in\mathbb{N}}, x) \to f(x)$ *is not computable.*

A practical representation for power series

Lemma

Let $f : \overline{B}(0,1) \to \mathbb{R}$ *be analytic and* $(a_n)_{n\in\mathbb{N}}$ *its power series around* 0.
Then there exists an $k, A \in \mathbb{N}$ *such that*

1. $\sqrt[k]{2}$ *is a lower bound on the radius of convergence*
2. $|a_n| \leq A \cdot 2^{-\frac{n}{k}}$

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A and k can be used to make a tail estimate on

$$\left| \sum_{n \geq N} a_n z^n \right|$$

Analytic Functions and Computational Complexity

Theorem (Kawamura, Rösnick, Müller, Ziegler (2013))

The following operators are computable in time polynomial in $n + k + \log(A)$, *where* 2^{-n} *is the output precision.*

1. *evaluation*
2. *addition and multiplication*
3. *differentiation and anti-differentiation*
4. *parametric maximization*

Ordinary Differential Equations

We are further interested in the following problem:
Consider Systems of Ordinary Differential Equations of the form

$$\begin{aligned} \dot{y}_v(t) &= F_v(t, y_1(t), \ldots, y_t(t)) \\ y_v(w_0) &= y_0 \end{aligned}$$

for $v = 1, \ldots, d$ and $F_v : \mathbb{R}^{d+1} \to \mathbb{R}$.

Complexity of Ordinary Differential Equations

Theorem (Kawamura, 2010)

Consider the IVP

$$y'(t) = f(t, y(t)) \quad ; \quad y(0) = 0.$$

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1. f *is Lipschitz-continuous and polynomial time computable*
2. y *is PSPACE-hard.*

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This is hard in the general case.
If all F_v are analytic

- the solutions y_v will again be analytic
- the coefficients of the power series can be computed in polynomial time if the F_v are polynomial time computable.

Multi-Dimensional Functions

Consider functions $f : \mathbb{R}^d \to \mathbb{R}$ analytic in a neighborhood of $0 \in \mathbb{C}^d$.

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The beforementioned representation can be generalized.
The main idea is let

$$f(x_1,\dots,x_d) = \sum_{i \in \mathbb{N}} b_i \cdot x_1^i$$

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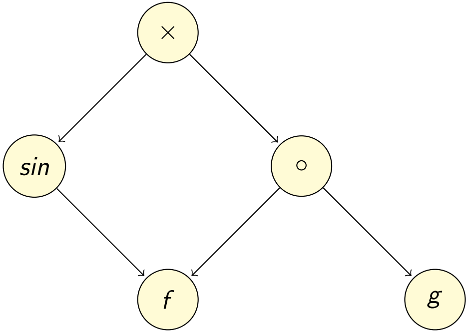
b_i can be evaluated by evaluating a $d-1$ dimensional analytic function.

DAG approach



Such a DAG can be evaluated or transformed to an analytic function.

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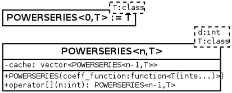

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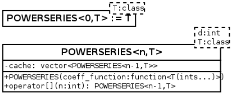

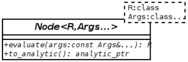

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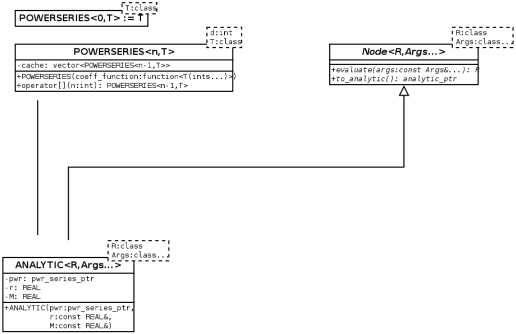

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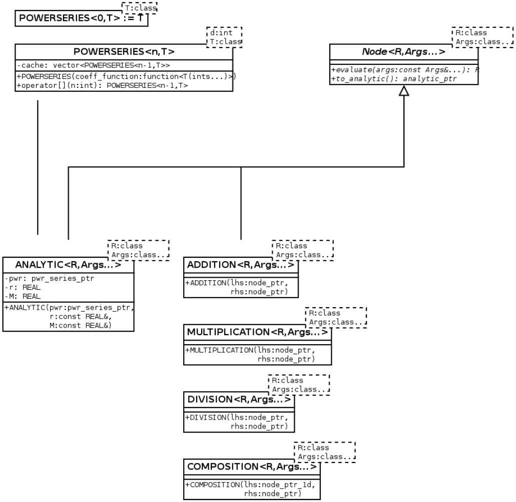

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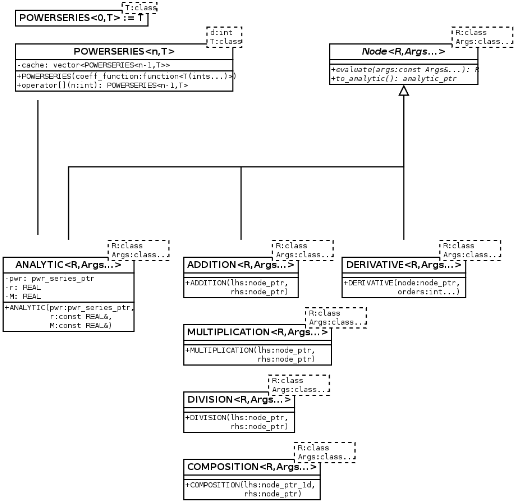

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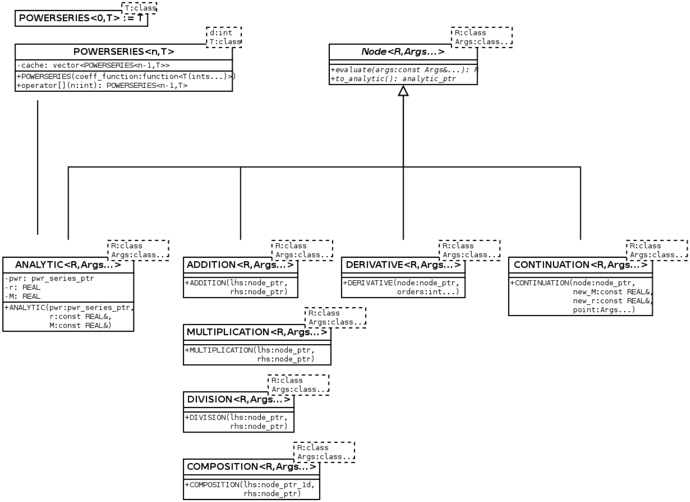

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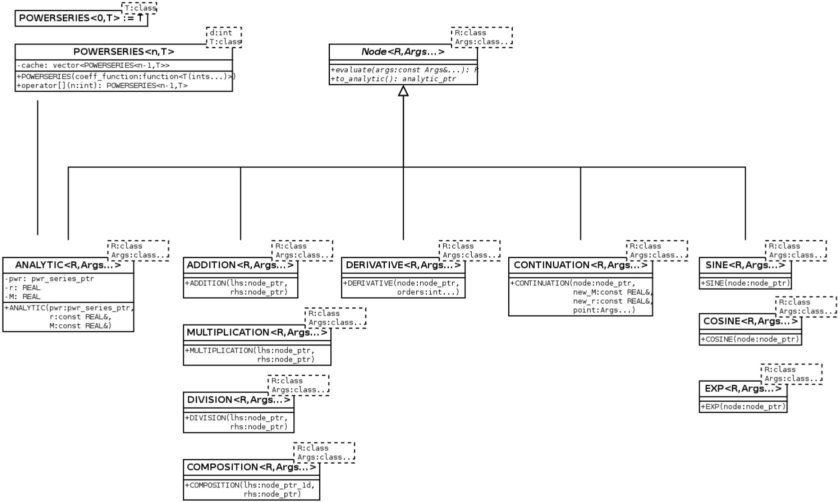

ODE solving

Additionally there is an operator `solve_ivp` for initial value problems of the form

$$\begin{aligned} \dot{y}_v(t) &= F_v(t, y_1(t), \ldots, y_t(t)) \\ y_v(0) &= 0 \text{ for } v = 1, \ldots, d \end{aligned}$$

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The solution has parameters $r' = \min(r, \frac{r}{M})$ $M' = r$

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- Solve the ODE $\dot{y}(t) = F(t, y(t))$; $y(t') = y_1$

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